

# Galilean Covariance and The Gravitational Field

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The paper is concerned with the development of a gravitational field theory having locally a covariant version of the Galilei group. We show that this Galilean gravity can be used to study the advance of perihelion of a planet, following in parallel with the result of the (relativistic) theory of general relativity in the post-Newtonian approximation.

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## I. INTRODUCTION

Since the birth of general relativity, several studies have been addressing the problem of an analogous (n-form) formulation for the non-relativistic theory of gravitation [1, 2, 3, 4, 5, 6, 7], which has as a fundamental structure the Galilei group. Beyond that, the interest in such a covariant description of the Galilei physics lies in the fact that some phenomena are restricted to Galilean regime. A known example is the superfluidity phenomenon, existing at low velocity, only [8]. Particularly in cosmology, in order to understand the large scale structures of the universe, the Newtonian gravity is required. Besides that, the rotation curve of galaxies is obtained with a Newtonian formalism [9], which was the first step to point to the existence of dark matter [10, 11, 12]. In a general sense, the Newtonian gravity theory is a natural choice to verify some insights on gravitation; thus if new ideas emerge from the development of general relativity, or even from a better theory than that, it has to be tested in order to reproduce the known results. In this sense, a geometric formulation of gravity based on the Galilei group (a Galilei gravity theory) may be of interest; and this is one of our goal here.

For such a purpose, it is important to develop a covariant form of Galilean transformations, since Galilei group acts as the symmetry group of Newtonian theory [2, 4]. This approach has been achieved by considering the space-time transformation in the light-cone of a 5-dimensional Minkowski space-time [4, 13, 14, 15, 16]. The 5-momentum vector is interpreted physically by considering 3 components describing the Euclidian momentum; one component standing for energy and the fifth component describing mass. In terms of space canon-

cal coordinates, one has three components for the space coordinates; one coordinate for time and the fifth component is associated to velocity. The consequence has been several developments for the non-relativistic classical and quantum field theory [13, 14, 15, 16, 17, 18, 19, 20, 21].

In this context of non-relativistic covariant physics, Duval et al [3] have addressed the problem of the gravitational field, using Bargmann structures, rather than Galilei group. In another direction, Carter and Chamel [5] adapted the procedures used in general relativity for application in purely Newtonian framework in order to provide new insights other than that of 3+1 decomposition of space-time. The formalism have been applied in the construction of a Newtonian fluid model to treat effects of superfluidity in neutrons stars [6, 7]. Here we follow a different perspective, avoiding the use of such a decomposition, since we work in the five dimensional space exploring the Galilei group. We develop a geometric description of a Galilean gravity parallelizing the usual general relativity. As an application we study the advance of perihelion of a planet and compare it to the results derived in the post-Newtonian version of the theory of general relativity.

Although our approach is quite close to the general relativity, they differ from each other by the fact that one is locally Lorentzian and the other is Galilean. The structure is the same and the equations will assume a similar tensor form, but the physical meaning is different. Since Galilei transformations are described in five dimensions as linear, similar to the Lorentz group, the tensor formalism developed in the context of general relativity, may be used. As a central result, our conclusions are genuine manifestations of the Galilean gravity and do not follow approximations of any kind of Einstein's equations, as it is the case, for example, in the post-Newtonian approximation [22], which is an expansion in terms of  $1/c$ , where  $c$  is the speed of light.

The paper is organized in the following way. In section 2, we set forth the notation and discuss briefly some aspects of the covariant formulation of Galilean transfor-

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mations. In section 3 we establish a geometrical formulation of Galilean gravity and in section 4, we apply it to the case of Schwarzschild-like line element to calculate the precession of the perihelion of a planet. In section 5, we present some concluding remarks.

## II. COVARIANT GALILEAN TRANSFORMATIONS

The Galilean transformations are given by

$$\begin{aligned} \mathbf{x}' &= R\mathbf{x} - \mathbf{V}t + \mathbf{a} \\ t' &= t + b, \end{aligned} \quad (1)$$

where  $R$  is a 3-dimensional Euclidian rotation,  $\mathbf{V}$  is the relative velocity defining the Galilei boost,  $\mathbf{a}$  stands for a space translations and  $b$  a time translation. In this realm of Galilean symmetries describing low velocities process, one can introduces a linear space-time tensor structure by noticing the following.

In non-relativistic physics, the dispersion relation for a free non-relativistic particle is given by  $E = \mathbf{p}^2/2m$ , where  $E$  is the energy,  $\mathbf{p}$  is the 3-dimensional momentum and  $m$  is the mass. This dispersion relation can also be written as

$$\mathbf{p}^2 - 2mH = 0. \quad (2)$$

Now let us consider the physical observable describing momentum as a quantity consisting of five entries, that is  $p^\mu = (\mathbf{p}, p^4, p^5)$ , where  $\mu = 1, \dots, 5$ ,  $\mathbf{p}$  is standing for the 3-vector momentum,  $p^4 = E/c'$  is the energy, and  $p^5 = c'm$  is mass. Here  $c'$  is a constant with units of velocity. We take  $c' = 1$ . Using this notation, and in order to recover Eq. (2), we write a general 5-dimensional dispersion relation, i.e.  $p_\mu p^\mu = p^\mu p^\nu \eta_{\mu\nu} = \mathbf{p}^2 - 2p^4 p^5 = k^2$ , where

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \quad (3)$$

This is taken as a metric tensor, that has been introduced in different ways in the literature, in particular, it was obtained in a (1+1) theory of gravitation by Cangemi and Jackiw [23].

Let us define the set of canonical coordinates associated to  $p^\mu$ , by writing a 5-vector in  $\mathcal{M}$  as  $q^\mu = (\mathbf{q}, q^4, q^5)$ . The entries in  $q^\mu$  are physically interpreted as follows:  $\mathbf{q}$  is the canonical coordinate attached to  $\mathbf{p}$ ;  $q^4$  is the canonical coordinate associated to  $E$ , and so it can be considered as the time coordinate;  $q^5$  is the canonical coordinate associated to  $m$ , and is explicitly given in terms of  $\mathbf{q}$  and  $q^4$  according to the corresponding dispersion relation, leading to  $q_\mu q^\mu = q^\mu q^\nu \eta_{\mu\nu} = \mathbf{q}^2 - 2q^4 q^5 = s^2$ . Since  $p_\mu p^\mu = 0$ , we have to take  $s = 0$ , leading to  $q^5 = \mathbf{q}^2/2t$ ;

or infinitesimally, we obtain  $\delta q^5 = \mathbf{v} \cdot \delta \mathbf{q}/2$ . Therefore the fifth component is basically defined by the velocity.

Let us study in more detail the content of canonical coordinates, by introducing a symplectic structure in the cotangent bundle  $T^*\mathcal{M}$ , through the 2-form  $\omega$ ,

$$\omega = \eta^{\mu\nu} dq_\mu \wedge dp_\nu, \quad \mu = 1, 2, \dots, 5. \quad (4)$$

Defining the vector field,

$$X_f = \frac{\partial f}{\partial p_\mu} \frac{\partial}{\partial q^\mu} - \frac{\partial f}{\partial q^\mu} \frac{\partial}{\partial p_\mu},$$

where  $f$  is a  $C^\infty$  function in the 10-dimensional (phase space) manifold  $\Omega$  with coordinates  $(q^\mu, p^\nu)$ , we have

$$\begin{aligned} \omega(X_f, X_h) &= \{f, h\} \\ &= dq(X_f)dp(X_h) - dp(X_f)dq(X_h), \\ &= \eta^{\mu\nu} \left( \frac{\partial f}{\partial q^\mu} \frac{\partial g}{\partial p^\nu} - \frac{\partial g}{\partial q^\mu} \frac{\partial f}{\partial p^\nu} \right), \end{aligned}$$

where  $\{f, g\}$  is the Poisson bracket. Observe that, since  $w(X_h) = dh$ , we have  $\{f, h\} = df(X_h) = \langle df, X_h \rangle$ .

Defining a flow by

$$\partial_\mu f = -X_{p_\mu} f, \quad (5)$$

where  $f(q, p)$  is a real ( $C^\infty$ ) density distribution function in  $\Omega$ , then, in terms of components, we have from Eq. (5),

$$\partial_i f = -X_{p_i} f \mapsto \partial_i f = \{p_i, f\}, \quad (6)$$

$$\partial_4 f = -X_{p_4} f \mapsto \partial_4 f = \{H, f\}, \quad (7)$$

$$\partial_5 f = -X_{p_5} f \mapsto \partial_5 f = 0. \quad (8)$$

Consistency relations are given by Eq. (6) and (8), while Eq. (7) describes the Liouville equation. In this way we recover the classical theory in the Liouville-Poisson representation.

Let us now turn our attention to the set of linear non-homogeneous transformations in  $\mathcal{M}$  of type  $\bar{q}^\mu = \Lambda^\mu_\nu q^\nu + a^\mu$ , leaving  $(dq^\mu dq'_\mu)$  invariant. In addition, we consider transformations connected to the identity, such that  $|\Lambda| = 1$ . In this case, for infinitesimal transformations we have  $\Lambda^\mu_\nu = \delta^\mu_\nu + \epsilon^\mu_\nu$ . Then we identify 15 generators of transformations. Using the definition

$$\hat{K}_\alpha = i \frac{\partial \bar{q}^\mu}{\partial \alpha} \bigg|_{\alpha=0} \frac{\partial}{\partial q^\mu}, \quad (9)$$

where  $K_\alpha$  is the generator associated with the group parameter  $\alpha$  (which also labels the group generators), we have

$$\hat{J}_3 = -i(q^1 \partial_2 - x^2 \partial_1), \quad (10)$$

$$\hat{J}_1 = -i(q^2 \partial_3 - q^3 \partial_2), \quad (11)$$

$$\hat{J}_2 = -i(q^3 \partial_1 - q^1 \partial_3), \quad (12)$$

$$\hat{G}_i = i(q^4 \partial_i + q^i \partial_5), \quad (13)$$

$$\hat{C}_i = i(q^5 \partial_i + q^i \partial_4), \quad (14)$$

$$\hat{D} = i(q^4 \partial_4 - q^5 \partial_5), \quad (15)$$

$$\hat{P}_\mu = i \partial_\mu, \quad (16)$$

where  $i = 1, 2, 3$  and  $\mu = 1, 2, \dots, 5$ . These generators satisfy the following commutation relations:

$$[\widehat{M}_{\mu\nu}, \widehat{M}_{\rho\sigma}] = -i[\eta_{\nu\rho}\widehat{M}_{\mu\sigma} - \eta_{\mu\rho}\widehat{M}_{\nu\sigma} + \eta_{\mu\sigma}\widehat{M}_{\nu\rho} - \eta_{\nu\sigma}\widehat{M}_{\mu\rho}], \quad (17)$$

$$[\widehat{P}_\mu, \widehat{M}_{\rho\sigma}] = -i[\eta_{\mu\rho}\widehat{P}_\sigma - \eta_{\mu\sigma}\widehat{P}_\rho], \quad (18)$$

$$[\widehat{P}_\mu, \widehat{P}_\nu] = 0, \quad (19)$$

where  $\widehat{M}_{\alpha\beta}$  ( $\alpha, \beta = 1, \dots, 5$ ) are defined by

$$\widehat{M}_{ij} = -\widehat{M}_{ji} = \varepsilon_{ijk}\widehat{J}_k, \quad (20)$$

$$\widehat{M}_{5i} = -\widehat{M}_{i5} = \widehat{G}_i, \quad (21)$$

$$\widehat{M}_{4i} = -\widehat{M}_{i4} = \widehat{C}_i, \quad (22)$$

$$\widehat{M}_{54} = -\widehat{M}_{45} = \widehat{D}. \quad (23)$$

The commutation relations given in Eqs. (17)–(19) is a Lie algebra, that we denote by  $\mathbf{g}$ . A subalgebra of  $\mathbf{g}$  is

$$\begin{aligned} [\widehat{L}_i, \widehat{L}_j] &= i\varepsilon_{ijk}\widehat{L}_k, \quad [\widehat{L}_i, \widehat{P}_j] = i\varepsilon_{ijk}\widehat{P}_k, \quad [\widehat{L}_i, \widehat{B}_j] = i\varepsilon_{ijk}\widehat{B}_k, \\ [\widehat{B}_i, \widehat{P}_4] &= i\widehat{P}_i, \quad [\widehat{B}_i, \widehat{P}_j] = i\widehat{P}_5\delta_{ij}, \end{aligned} \quad (24)$$

corresponding to the Galilei-Lie algebra with the usual central charge  $\widehat{P}_5$  describing mass. Notice that here the central charge arises naturally from the isometry in 5-dimensions.

The dispersion relation  $p_\mu p^\mu = 0$  defines a Galilean vector in the light-cone. However, in a more general case we have

$$\begin{aligned} p_\mu p^\mu &= p^2 - 2mE = k^2 \\ E + k^2/2m &= \frac{p^2}{2m}. \end{aligned} \quad (25)$$

The constant  $k^2$  is absorbed into the energy by means of the definition  $E' = E + k^2/2m$ . Therefore, we recover the dispersion relation  $E' = p^2/2m$ , which is physically consistent. Then we can work with  $k \neq 0$ .

### III. GEOMETRIC APPROACH TO GALILEAN GRAVITY

In order to generalize this formalism to a curved Galilean space-time, we introduce Galilean tensor. Considering

$$\frac{\partial x^{\mu'}}{\partial x^\nu} = \Lambda^\mu{}_\nu, \quad (26)$$

where  $\Lambda^\mu{}_\nu$  is given explicitly by

$$\begin{pmatrix} \mathbf{x}' \\ x^{4'} \\ x^{5'} \end{pmatrix} = \begin{pmatrix} R & 0 & -\mathbf{V} \\ -\mathbf{V} \cdot R & 1 & \frac{1}{2}V^2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ x^4 \\ x^5 \end{pmatrix}, \quad (27)$$

we define covariant and contravariant components of tensors as usual. Taking a non-flat manifold where locally the metric is  $\eta$ , we define a covariant derivative as

$$\nabla_\mu X^\nu = \partial_\mu X^\nu + \Gamma_{\lambda\mu}^\nu X^\lambda, \quad (28)$$

where  $\Gamma_{\lambda\mu}^\nu$  is a connection that stipulates the nature of Galilean space-time. The covariant derivative of a scalar reduces to the normal derivative.

Let us give a definition for the curvature tensor. The current idea to define curvature resides on an intuitive concept. If we consider a vector field  $X^\mu$  on a closed circuit on a manifold and if there is any change in direction of  $X^\mu$  after a round around the circuit then we say that this manifold is curved. Mathematically it is stated as

$$\nabla_{[\mu} \nabla_{\lambda]} X^\nu = \frac{1}{2} R^\nu{}_{\gamma\mu\lambda} X^\gamma, \quad (29)$$

where  $R^\nu{}_{\gamma\mu\lambda}$  is the curvature tensor. We assume that when there is no gravitational field, the curvature tensor vanishes.

The metric tensor is a covariant tensor of rank 2. It is used to define distances and lengths of vectors. The infinitesimal distance between two points  $x^a$  and  $x^a + dx^a$  in curved manifold defined from  $\mathcal{M}$  is defined by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (30)$$

where  $g_{\mu\nu}$  is the metric tensor. The relation (30) represents the line element as well. We have to notice that the metric  $\eta_{\mu\nu}$  in (3) defines a flat line element. The imposition that the covariant derivative of metric is zero yields the following expression for the connection

$$\Gamma_{\lambda\mu}^\nu = \frac{1}{2} g^{\nu\delta} (\partial_\lambda g_{\delta\mu} + \partial_\mu g_{\delta\lambda} - \partial_\delta g_{\lambda\mu}). \quad (31)$$

When the connection is written as in Eq. (31) the manifold is said to be an affine manifold.

The curvature tensor defined in affine manifold has the following properties:

$$\begin{aligned} R_{\mu\nu\lambda\gamma} &= -R_{\mu\nu\gamma\lambda} = -R_{\nu\mu\lambda\gamma} = R_{\lambda\gamma\mu\nu} \\ R_{\mu\nu\lambda\gamma} + R_{\mu\gamma\nu\lambda} + R_{\mu\lambda\gamma\nu} &\equiv 0. \end{aligned} \quad (32)$$

These properties are derived from Eq. (29). If we perform a contraction of the indices of the curvature tensor then it is possible to define the Galilei-invariant curvature scalar

$$R = g^{\mu\nu} g^{\gamma\lambda} R_{\gamma\mu\lambda\nu}. \quad (33)$$

To generate the field equations, we write a Lagrangian invariant under Galilean transformations. A natural candidate is the curvature scalar, then the action in a general form is

$$I = \int_\Omega d\Omega (\sqrt{-g} R + k L_m), \quad (34)$$

where  $g = \det g_{\mu\nu}$ ,  $k$  is the coupling constant,  $L_m$  is a matter lagrangian density and  $d\Omega$  is the 5-dimensional

element of volume. Varying the action with respect to  $g_{\mu\nu}$ , we obtain

$$\begin{aligned}\frac{\delta(\sqrt{-g}R)}{\delta g_{\mu\nu}} &= -\sqrt{-g}(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R), \\ \frac{\delta L_m}{\delta g_{\mu\nu}} &= \sqrt{-g}T^{\mu\nu},\end{aligned}\quad (35)$$

where the latter equation defines the energy-momentum tensor of matter fields and  $R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$ . Thus the field equation becomes

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = kT^{\mu\nu}. \quad (36)$$

These equations have the same form as for the General Relativity equations, since the Galilean transformations are written in a way similar to Lorentz transformations. Let us note that, the quantity  $k$  is just a coupling constant between the Galilean gravity and matter fields (it has nothing to do with Einstein's constant).

We have to note that the equations in (36) with  $\mu = 4$ , i.e. the equations

$$R^4_{\nu} - \frac{1}{2}\delta^4_{\nu}R = kT^4_{\nu}, \quad (37)$$

contain only the first order derivative of the components of  $g_{\mu\nu}$  with respect to time. Actually in (37) the components of the form  $R_{4i4j}$  drop out, where the indices  $i$  and  $j$  run from 1 to 3 and assume the value 5 as well. In face of this we note that some components of time derivative of the metric tensor are associated with the freedom of the choice of the system of coordinates. So we have to specify in a particular coordinate system only the time derivatives of  $g_{ij}$  as initial conditions. Therefore we see the ten equations

$$R^i_j - \frac{1}{2}\delta^i_j R = kT^i_j, \quad (38)$$

where the indices  $i$  and  $j$  run from 1 to 3 and assume the value 5 as well, as dynamical equations and the five equations (37) as constraint equations.

#### IV. THE SCHWARZSCHILD-LIKE LINE ELEMENT: THE ADVANCE OF THE PERIHELION OF A PLANET

Since the metric determines every feature of a system described by a geometrical approach, we intend to get a spherically symmetric metric [24]. Then we introduce a Galilean Schwarzschild solution. The Galilean Schwarzschild line element is defined by

$$ds^2 = f^{-1}(r)dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 - f(r)dsdt, \quad (39)$$

where as usual  $f = 1 - 2M/r$ . This metric describes a system with spherical symmetry. In the following we

consider this line element to study the movement of a planet.

The space-time in the exterior region of a massive body can be described by the line element in Eq. (39). For example, for the system composed of Sun and Mercury, only force acting on this system is the gravitational one. Therefore, the movement will be geodesic. A two-body system can be described by means of the reduced mass as a consequence we deal with a one body system. We perform the calculations considering the mass in the line element as the reduced mass.

The geodesic movement can be described by means a variational principle where the action is the interval between two events in space-time. Then the equation of movement is given by Euler-Lagrange equation, i.e.

$$\begin{aligned}\delta s &= \int \delta ds = \int \delta L d\tau \\ &= \int \delta(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)^{1/2} d\tau = 0,\end{aligned}\quad (40)$$

where the dot represents the derivative with respect to the proper time  $\tau$ . The meaning of proper time remain the same of that defined in General Relativity, once our theory share the same feature with respect to transformation of coordinates. In this context the proper time is

$$\tau = \int (-g_{00})^{1/2} dt, \quad (41)$$

where the coordinate  $t$  could not assume necessarily the meaning of time. Thus the meaning of each coordinate depends on which system of coordinates one is using.

Instead of working with  $L = (g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)^{1/2}$ , we consider the quantity defined by

$$K = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu, \quad (42)$$

which obey the Euler-Lagrange equation as well. Of course  $K$  can be set equal to a constant which takes the possible values 1, -1 or 0. This relation is completely similar to that given in Eq. (25) defined on the flat Galilean space-time. If we perform the sum in Eq. (42), the expression assumes the following form

$$f^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2 - 2f\dot{s} = \alpha. \quad (43)$$

If we put  $K$  into the Euler-Lagrange equation and consider this movement restricted to a plane ( $\theta = \pi/2$ ), since the angular momentum is a constant, then we get for  $\mu = (3, 4, 5)$  the following equations, respectively,

$$r^2\dot{\phi} = h, \quad f\dot{s} = \beta, \quad f\dot{t} = \kappa. \quad (44)$$

These equations represent conservation laws. In fact, the quantities  $\kappa$ ,  $\beta$  and  $h$  are related to energy, mass and angular momentum respectively. The relations given in

Eq. (44) can be substituted into Eq. (43) with  $\theta = \pi/2$ , leading to

$$f^{-1}\dot{r}^2 + \frac{h^2}{r^2} - 2\frac{\beta\kappa}{f} = \alpha. \quad (45)$$

At this point we find an equation for the trajectory by changing the variable in Eq. (45). We define  $U = U(\phi) = \frac{1}{r}$ ; such that the derivative of  $U$  with respect to  $\phi$ , which will be designated by  $U'$ , is equal to  $h$  times the derivative of  $r$  with respect to the proper time. The final equation is,

$$U'^2 + fU^2 - 2\frac{\kappa\beta}{h^2} = \frac{\alpha f}{h^2}. \quad (46)$$

Taking the derivative of the above equation with respect to  $\phi$  and remembering that  $f = 1 - 2M/r$ , we obtain the trajectory equation,

$$U'' + U = 3MU^2 - \frac{\alpha M}{h^2}. \quad (47)$$

Choosing  $\alpha = -1$ , we have

$$U'' + U = 3MU^2 + \frac{M}{h^2}, \quad (48)$$

which is the same equation obtained in General Relativity. In this context, however, there does not exist a relation of causality due to the constancy of velocity of light, since there is no such imposition in a Galilean theory. As a consequence, the advance of perihelion of a planet is given by the usual expression [25]

$$\delta\phi = 6\pi \frac{M^2}{h^2}. \quad (49)$$

It is important to rewrite Eq. (49) with the constants  $G$  and  $c'$ . In this case we have

$$\delta\phi = 6\pi \frac{G^2 M^2}{c'^2 h^2} = \frac{24\pi^3 a^2}{c'^2 T^2 (1 - e^2)}, \quad (50)$$

where  $c'$  is a typical velocity of the system to be fixed experimentally for the Galilean symmetry,  $M$  is the reduced mass,  $T$  is the period of the movement,  $e$  is the eccentricity of the orbit and  $a$  is semi-major axis of the ellipse. We have to note that  $\delta\phi$  is dimensionless.

In order to get the well known Newtonian equation for the planetary movement under the influence of a force proportional to the inverse of square radius, we have to rewrite Eq. (48) with the constants  $G$  and  $c'$ , such that

$$U'' + U = G \frac{M}{h^2} + 3 \frac{G}{c'^2} MU^2. \quad (51)$$

This assumes the post-Newtonian equation for the planetary movement, up to the second term on right-hand-side of above equation. This term is a correction of the classical equation and the parameter  $c'$  can be taken experimentally. Therefore the post-Newtonian equation arises from the 5-dimensional Galilean gravity if we consider the parameter to adjust dimensionality greater than the other parameters in Eq. (51); thus establishing a weak field regime. It is important to notice that Eq. (50) gives a precession of  $43''/\text{century}$  for planet Mercury, since the parameter  $c'$  is taken to be velocity of light  $c$ . The consistence of this choice can be established by noticing that, taking the Lorentz definition of energy and the limit of  $v/c$  much less than 1, then energy and momentum are related by  $E = p^2/(2m) + mc^2$ , corresponding to a dispersion relation as given in Section 2,  $p_\mu p^\mu = p^\mu p^\nu \eta_{\mu\nu} = \mathbf{p}^2 - 2p^4 p^5 = k^2$ , with  $k^2 = 2(mc)^2$ .

## V. CONCLUSION

We have established a geometric theory of gravity in the framework of Galilean symmetry. These transformations are taken in a covariant form and the flat space-time is defined with the Galilean metric. With curved space-time we have associated this to the gravitation, in parallel to the usual relativistic case. Then we construct a Schwarzschild-like line element, which is spherically symmetric, and apply it to the movement of a planet. As a result, we derive the post-Newtonian result of the general relativity in a covariant form. We have shown that our equation assumes the Newtonian form in the weak field regime. One basic physical new fact to be learned from all this is that, the advance of perihelion of a planet is a geometric effect that can be described fully in a Galilean covariant theory.

It would be of interest to analyze other related problems in this context of Galilean gravity, as for instance the bending of light, other gravitation models such as cosmological models [26], the hamiltonian approach which could reveal more about the structure of field equations and a Galilean gauge theory. These aspects will be discussed elsewhere.

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- [1] P. Havas, Rev. Mod. Phys. **36**, 938 (1964).  
 [2] H. P. Kunzle, Gen. Rel. Grav. **7**, 445 (1976).

- [3] C. Duval, G. Burdet, H. P. Kunzle, and M. Perrin, Phys. Rev. D **31**, 1841 (1985).

- [4] H. P. Kunzle and C. Duval, *Class. Quant. Grav.* **3**, 957 (1986).
- [5] B. Carter and N. Chamel, *Int. J. Mod. Phys. D* **13**, 291 (2004), astro-ph/0305186.
- [6] B. Carter and N. Chamel, *Int. J. Mod. Phys. D* **14**, 717 (2005), astro-ph/0312414.
- [7] B. Carter and N. Chamel, *Int. J. Mod. Phys. D* **14**, 749 (2005), astro-ph/0410660.
- [8] L. D. Landau and E. M. Lifshiz, *Fluid Mechanics*, vol. 6 of *Course of Theoretical Physics* (Elsevier Butterworth-Heinemann, 2004).
- [9] Y. Brihaye, C. Gonera, S. Giller, and P. Kosinski (1995), hep-th/9503046.
- [10] M. Persic, P. Salucci, and F. Stel, *Mon. Not. Roy. Astron. Soc.* **281**, 27 (1996), astro-ph/9506004.
- [11] L. D. Matthews and J. S. Gallagher III, *The Astrophysical Journal Supplement Series* **141**, 429 (2002).
- [12] S. M. Faber and J. S. Gallagher, *Ann. Rev. Astron. Astrophys.* **17**, 135 (1979).
- [13] Y. Takahashi, *Fortschr. Phys.* **36**, 63 (1988).
- [14] Y. Takahashi, *Fortschr. Phys.* **36**, 83 (1988).
- [15] M. Omote, et al, *Fortschr. Phys.* **37**, 933 (1989).
- [16] A.E. Santana, F.C. Khanna, Y. Takahashi, *Prog. Theor. Phys.* **99**, 327 (1998).
- [17] M. de Montigny, F.C. Khanna, A.E. Santana, E.S. Santos, J.D.M. Vianna, *Ann. Phys. (N.Y.)* **277**, 144 (1999).
- [18] M. de Montigny, F.C. Khanna, A.E. Santana, E.S. Santos, J.D.M. Vianna, *J. Phys. A: Math. Gen.* **33**, L273 (2000).
- [19] M. de Montigny, F. C. Khanna, A. E. Santana, and E. S. Santos, *J. Phys. A: Math. Gen.* **34**, 8901 (2001).
- [20] E. S. Santos, M. de Montigny, F. C. Khanna, and A. E. Santana, *J. Phys. A: Math. Gen.* **37**, 9771 (2004).
- [21] M. Kobayashi, M. de Montigny, and F. C. Khanna, *Phys. Lett. A* **372**, 3541 (2008).
- [22] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley, N. York, 1972).
- [23] D. Cangemi and R. Jackiw, *Ann. Phys. (N.Y.)* **225**, 229 (1993).
- [24] R. d’Inverno, *Introducing Einstein’s Relativity* (Clarendon Press, Oxford, 1996).
- [25] L. D. Landau and E. M. Lifshiz, *The Classical Theory of Fields*, vol. 2 of *Course of Theoretical Physics* (Elsevier Butterworth-Heinemann, 2004).
- [26] R. K. Sachs, A. M. Wolfe, G. Ellis, J. Ehlers, and A. Krasinski, *Gen. Rel. Grav.* **39**, 1929 (2007).